MATH-4800 **NUMERICAL COMPUTING** **Summer 2023**

Assignment 1, due before class, Wednesday May 31, 2023.

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1. Write a `Matlab` code to evaluate a polynomial using Horner's method as described on page 3 of the text (and discussed in class). Use your code to evaluate the polynomials in Text Exercises $1(b), 2(c)$ and $4(a)$ on page 5.

```
%Program 0.1 Nested multiplication
%Evaluates polynomial from nested form using Horner's Method
%Input: degree d of polynomial,
%       array of d+1 coefficients c (constant term first),
%       x-coordinate x at which to evaluate, and
%       array of d base points b, if needed
%Output: value y of polynomial at x
function y=nest(d,c,x,b)
if nargin<4, b=zeros(d,1); end
y=c(d+1);
for i=d:-1:1
  y = y.*(x-b(i))+c(i);
end
```

```
>> nest(4,[1 -5 5 4 -3],1/3)

ans =

     0

>> nest(6,[4 -2 0 0 -2 0 4],-1/2)

ans =

    4.9375

>> nest(3,[1 1/2 1/2 -1/2],5,[0 2 3])

ans =

    -4
```

2. Text Exercises 1(a) and 1(b) on page 20, and Computer Problem 1 on page 20.

$\frac{1-\sec x}{\tan^2 x}$: There is subtraction of nearly equal numbers when $1 - \sec x \approx 0$. Setting $\sec x = 1$, we have $\boxed{x = 2k\pi}$ for $k \in \mathbb{Z}$. Reformulating by using the trig identity $\sec^2 x = 1 + \tan^2 x$, we have that

$$\frac{1-\sec x}{\tan^2 x} \cdot \frac{1+\sec x}{1+\sec^2 x} = \frac{\cancel{1-\sec^2 x}\ -\tan^2 x}{\tan^2 x(1+\sec x)} = \boxed{-\frac{1}{1+\sec x}}$$

avoiding the cancellation problem for x near $2k\pi$.

$\frac{1-(1-x)^3}{x}$: There is substraction of nearly equal numbers when $1-(1-x)^3 \approx 0$. Setting $(1-x)^3 = 1$, we have $\boxed{x = 0}$. Simplifying the expression yields

$$\frac{1-(1-x)^3}{x} = \frac{1-(1-3x+3x^2-x^3)}{x}$$

$$= \frac{x^3 - 3x^2 + 3x}{x}$$

$$= \boxed{x^2 - 3x + 3}$$

avoiding the cancellation problem for x near 0.

x	original $\frac{1-\sec x}{\tan^2 x}$	revised $-\frac{1}{1+\sec x}$
0.10000000000000	−0.49874791371143	−0.49874791371143
0.01000000000000	−0.49998749979096	−0.49998749979166
0.00100000000000	−0.49999987501429	−0.49999987499998
0.00010000000000	−0.49999999362793	−0.49999999875000
0.00001000000000	−0.50000004133685	−0.49999999998750
0.00000100000000	−0.50004445029084	−0.49999999999987
0.00000010000000	−0.51070259132757	−0.50000000000000
0.00000001000000	0	−0.50000000000000
0.00000000100000	0	−0.50000000000000
0.00000000010000	0	−0.50000000000000
0.00000000001000	0	−0.50000000000000
0.00000000000100	0	−0.50000000000000
0.00000000000010	0	−0.50000000000000
0.00000000000001	0	−0.50000000000000

x	original $\frac{1-(1-x)^3}{x}$	revised x^2-3x+3
0.10000000000000	2.71000000000000	2.71000000000000
0.01000000000000	2.97010000000001	2.97010000000000
0.00100000000000	2.99700100000000	2.99700100000000
0.00010000000000	2.99970000999905	2.99970001000000
0.00001000000000	2.99997000008379	2.99997000010000
0.00000100000000	2.99999700015263	2.99999700000100
0.00000010000000	2.99999969866072	2.99999970000001
0.00000001000000	2.99999998176759	2.99999997000000
0.00000000100000	2.99999991515421	2.99999999700000
0.00000000010000	3.00000024822111	2.99999999970000
0.00000000001000	3.00000024822111	2.99999999997000
0.00000000000100	2.99993363483964	2.99999999999700
0.00000000000010	3.00093283556180	2.99999999999970
0.00000000000001	2.99760216648792	2.99999999999997

3. Let $\tilde{f}(x)$ approximate $f(x)$, where

$$f(x) = \frac{1}{\sqrt{3+x}}, \qquad \tilde{f}(x) = \frac{1}{2} - \frac{x-1}{16}.$$

Find the absolute forward and backward errors in the approximation when

(a) $x = 0.9$,

(b) $x = 1.2$.

(a) Compute $f(0.9) = \dfrac{1}{\sqrt{3+0.9}} = 0.50637$

$\hat{f}(0.9) = \dfrac{1}{2} - \dfrac{0.9-1}{16} = 0.50625 = \hat{y}$

Forward absolute error $= |\hat{f}(0.9) - f(0.9)| = \boxed{0.00012}$. //

Let $\hat{x}$ solve $\hat{y} = f(\hat{x})$, then $0.50625 = \dfrac{1}{\sqrt{3+\hat{x}}}$

$\hat{x} = \dfrac{1}{(0.50625)^2} - 3$

$= 0.90184$

Backward absolute error $= |\hat{x} - x| = \boxed{0.00184}$. //

(b) Compute $f(1.2) = \dfrac{1}{\sqrt{3+1.2}} = 0.48795$

$$\hat{f}(1.2) = \frac{1}{2} - \frac{1.2-1}{16} = 0.48750 = \hat{y}$$

Forward absolute error $= |\hat{f}(1.2) - f(1.2)| = \boxed{0.00045}$.

Let $\hat{x}$ solve $\hat{y} = f(\hat{x})$, then $0.48750 = \dfrac{1}{\sqrt{3+\hat{x}}}$

$$\hat{x} = \frac{1}{(0.48750)^2} - 3$$

$$= 1.207758$$

Backward absolute error $= |\hat{x} - x| = \boxed{0.007758}$.

4. The roots x_1 and x_2 of the quadratic equation $x^2 + 2bx + c = 0$ may be computed using the results of the usual quadratic formula

$$x_1 = -b - \sqrt{b^2 - c}, \qquad x_2 = -b + \sqrt{b^2 - c}.$$

(a) Use 6-digit, base 10, floating-point arithmetic to compute the two roots using the formulas above for the cases (i) $b = 1.23456 \times 10^6, c = 9.87654 \times 10^8$ and (ii) $b = -2.46864 \times 10^{-2}, c = 1.35753 \times 10^{-8}$. Compute the "exact" values using full precision on a calculator or using `Matlab`. Determine the relative (forward) errors in the roots computed using 6-digit arithmetic. Are the 6-digit roots accurate? Explain the results in terms of round-off error.

(b) Write a `Matlab` function, called `myRoots` say, that takes as input the values of b and c in the quadratic equation above and returns x_1 and x_2 computed accurately. Base the calculation of the roots in your Matlab function on the formulas for x_1 and x_2 above, or the alternate forms.

$$x_1 = \frac{c}{-b + \sqrt{b^2 - c}}, \qquad x_2 = \frac{c}{-b - \sqrt{b^2 - c}}.$$

Your `Matlab` function should check the input values and decide which formulas give the most accurate results. Run your code for the two cases given in part (a).

(a) Quadratic formula $\Rightarrow$ $x_1 = -b - \sqrt{b^2 - c}$

$x_2 = -b + \sqrt{b^2 - c}$

(i) $b = 1.23456 \times 10^6$, $c = 9.87654 \times 10^8$

use the result of rounding in the next step till the end of computation $\Rightarrow$

$(b^2)_c = 1.52414 \times 10^{12}$

$(b^2 - c)_c = 1.52315 \times 10^{12}$

$(\sqrt{b^2 - c})_c = 1.23416 \times 10^6$

$$\boxed{\begin{aligned} (x_1)_c &= -2.46872 \times 10^6 \\ x_1 &= -2.46872 \times 10^6 \\ r_1 &= 0 \\ (x_2)_c &= -4 \times 10^2 \\ x_2 &= -400.067 \\ r_2 &\approx 1.67 \times 10^{-4} \end{aligned}}$$

//

Quadratic formula $\Rightarrow$ $x_1 = -b - \sqrt{b^2 - c}$

$x_2 = -b + \sqrt{b^2 - c}$

(ii) $b = -2.46864 \times 10^{-2}$, $c = 1.35753 \times 10^{-8}$

$$(b^2)_c = 6.09418 \times 10^{-4}$$

$$(b^2 - c)_c = 6.09404 \times 10^{-4}$$

$$(\sqrt{b^2 - c})_c = 2.46861 \times 10^{-2}$$

$$\boxed{\begin{aligned} (x_1)_c &= 3 \times 10^{-7} \\ x_1 &= 2.74957 \times 10^{-7} \\ r_1 &\approx 9.11 \times 10^{-2} \\ (x_2)_c &= 4.93725 \times 10^{-2} \\ x_2 &= 4.93725 \times 10^{-2} \\ r_2 &= 0 \end{aligned}}$$

//

Nonzero errors occur when we add two numbers with opposite sign : cancellation of the significant digits leaves the accumulation of the previous steps' round-off error as the result.

```
function x=myRoots(b,c)
%
% compute distinct real roots of the quadratic x^2+2*b*x+c=0
% using the given quadratic formula or the alternative
% based on the input value b
%

% calculate the discrimiant
discriminant=b^2-c;

% check the discriminant to make sure the roots are real
if discriminant<=0
    disp('Error: discriminant must be positive')
    x=[0 0];
    return
end

% By using the result from part A %
%
% if b positive
% -b and -sqrt(discriminant) have the same sign
% -b and sqrt(discriminant) have opposite sign
% thus we find x1 using the given formula
% and find x2 using the alternative
%
if b>0
    x1=-b-sqrt(discriminant);
    x2=c/x1;
%
% if b negative
% -b and -sqrt(discriminant) have opposite sign
% -b and sqrt(discriminant) have the same sign
% thus we find x2 using the given formula
% and find x1 using the alternative
%
else
    x2=-b+sqrt(discriminant);
    x1=c/x2;
end

% save the roots in the vector x
x=[x1 x2];
```

compare with results from part (a)

```
>> b=1.23456*10^6; c=9.87654*10^8;
>> [x1,x2]=myRoots(b,c)

x1 =

  -2.4687e+06


x2 =

 -400.0673


>> b=-2.46864*10^-2; c=1.35753*10^-8;
>> [x1,x2]=myRoots(b,c)

x1 =

   2.7496e-07


x2 =

    0.0494
```

5. Text Exercise 6 on page 24.

(a) $$P_4(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}f'''(x_0)(x-x_0)^3 + \frac{1}{24}f^{(4)}(x_0)(x-x_0)^4$$

$\Rightarrow$ $f(x) = x^{-2}$ and $x_0 = 1$

The first four derivatives of $f(x) = x^{-2}$ are

$f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, $f^{(4)}(x) = 120x^{-6}$.

At $x_0 = 1$, these derivatives are

$f(1) = 1$, $f'(1) = -2$, $f''(1) = 6$, $f'''(1) = -24$, $f^{(4)}(1) = 120$.

Substituting them into the formula for $P_4(x)$ yields

$$\boxed{P_4(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4}.$$

//

(b) $$P_4(0.9) = 1 - 2(0.9-1) + 3(0.9-1)^2 - 4(0.9-1)^3 + 5(0.9-1)^4$$

$$= \boxed{1.2345}.$$

//

$$P_4(1.1) = 1 - 2(1.1-1) + 3(1.1-1)^2 - 4(1.1-1)^3 + 5(1.1-1)^4$$

$$= \boxed{0.8265}.$$

//

(c) By the Lagrange's form of the remainder, the error bound is given by $\frac{6|x-1|^5}{c^7}$. To maximize error, we minimize the denominator by setting $c = 0.9$. Thus, for $x = 0.9$, the error is bounded by $\boxed{1.25445 \times 10^{-4}}$, and for $x = 1.1$, the error bound remains the same.

We expect the approximation to be $\boxed{\text{better for } x > 1}$ because the derivatives increase greatly as $x \to 0$.

(d) Actual error for $x = 0.9$: $1.234567901 - 1.2345$

$= 6.7901 \times 10^{-5}$ smaller than

$< 1.25445 \times 10^{-4}$ the error bound

Actual error for $x = 1.1$: $0.8265 - 0.82644628$

$= 5.372 \times 10^{-5}$ smaller than

$< 1.25445 \times 10^{-4}$ the error bound