MATH 2400 EXAM 2 SPRING 2023 Name Haowen He Section

(3 Multiple Choice Questions) 4.5pts + (3 Questions) 15.5pts + (1 Question) 1 bonus pt

Multiple Choice Questions

1. **(1.5 pts)** Given A = $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$. Which of the following statement is true?

   a. A is defective. $\det(A) = 1\cdot(-2) - (-1)\cdot 2 = 0$
   b. A is chaotic.
   c. A is singular.
   d. None of the above.

   answer: c

2. **(1.5 pts)** Which of the following pair of vectors is linearly independent? Given *a, b* are arbitrary constants.

   a. $\begin{pmatrix} a \\ 2a \end{pmatrix}, \begin{pmatrix} 3\pi \\ 6\pi \end{pmatrix}$
   b. $\begin{pmatrix} ab \\ 2a \end{pmatrix}, \begin{pmatrix} ab^2 \\ 2ab \end{pmatrix}$
   c. $\begin{pmatrix} a \\ 2a \end{pmatrix}, \begin{pmatrix} 3b \\ b \end{pmatrix}$
   d. $\begin{pmatrix} 2\pi \\ \pi e \end{pmatrix}, \begin{pmatrix} 2 \\ e \end{pmatrix}$

   answer: c

3. **(1.5 pts)** Given the solution of a linear system be $\boldsymbol{y} = \begin{pmatrix} 3C_1e^{2t} + 2C_2e^{-t} \\ C_1e^{2t} - C_2e^{-t} \end{pmatrix}$. This system is

   a. Asymptotically stable $\lambda_1 = 2$ & $\lambda_2 = -1$
   b. Unstable
   c. Neutrally stable saddle point
   d. None of the above

   $\Downarrow$

   unstable

   answer: b

4. (5 pts) Find the general solution of $2x^2y'' + 3xy' - y = x$, $y_1 = x^{1/2}$ by ***reduction of order***.

let $y = y_1 v = x^{1/2} v$

$y' = \frac{1}{2}x^{-1/2}v + x^{1/2}v'$

$y'' = -\frac{1}{4}x^{-3/2}v + x^{-1/2}v' + x^{1/2}v''$

equation becomes

$$2x^2\left(\cancel{-\frac{1}{4}x^{-3/2}v} + x^{-1/2}v' + x^{1/2}v''\right) + 3x\left(\cancel{\frac{1}{2}x^{-1/2}v} + x^{1/2}v'\right) - \cancel{x^{1/2}v} = x$$

$2x^{5/2}v'' + 5x^{3/2}v' = x$

let $u = v'$

$u' + \frac{5}{2}x^{-1}u = \frac{1}{2}x^{-3/2}$

let $\mu = e^{\frac{5}{2}\int 1/x \, dx}$

$= x^{5/2}$

$(x^{5/2}u)' = x^{5/2} \cdot \frac{1}{2}x^{-3/2}$

$= \frac{1}{2}x$

$x^{5/2}u = \frac{1}{4}x^2 + c_1$

$u = \frac{1}{4}x^{-1/2} + c_1 x^{-5/2}$

$v = \int u \, dx$

$= \frac{1}{2}x^{1/2} - \cancel{\frac{2}{3}c_1}^{\,c_3} x^{-3/2} + c_2$

$y = x^{1/2}v$

$= \boxed{\frac{1}{2}x - c_3 x^{-1} + c_2 x^{1/2}}$

5. (5.5 pts) Given a system $\boldsymbol{y}' = \begin{pmatrix} 3 & -4 \\ 0 & 3 \end{pmatrix} \boldsymbol{y}$, with the matrix $A = \begin{pmatrix} 3 & -4 \\ 0 & 3 \end{pmatrix}$ being defective.

a. (2 pts) Find the corresponding eigenvalues , λ, and the first eigenvector, $\boldsymbol{x}^{(1)}$.
b.(3.5 pts) Assume $\boldsymbol{y}^{(2)} = \boldsymbol{x}^{(2)} e^{\lambda t}$, where $\boldsymbol{x}^{(2)} = t\boldsymbol{x}^{(1)} + \boldsymbol{\varphi}$ and $(A - \lambda I)\boldsymbol{\varphi} = \boldsymbol{x}^{(1)}$. Find the particular solution of the system, where $\boldsymbol{y}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

a) $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -4 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$\lambda = 3$ (repeated)

$\lambda = 3$, $(A - 3I)\vec{x} = 0$

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = a$; $x_2 = 0$

$$\vec{x} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

b) $(A - 3I)\vec{\varphi} = \vec{x}^{(1)}$

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\varphi_1 = a$; $\varphi_2 = -\frac{1}{4}$

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} a \\ -1/4 \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$$

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}$$

$$+ c_2\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}\right) e^{3t}$$

$$= \begin{pmatrix} c_3 + c_2 t \\ -\frac{1}{4}c_2 \end{pmatrix} e^{3t}$$

$$\vec{y}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} c_3 \\ -\frac{1}{4}c_2 \end{pmatrix}$$

$\Rightarrow$ $c_2 = -12$;

$c_3 = 2$

$$\Rightarrow \boxed{\vec{y} = \begin{pmatrix} 2 - 12t \\ 3 \end{pmatrix} e^{3t}}$$

6. (5 pts) Given the nonlinear system:

$$\frac{dx}{dt} = -(2+y)(x+y) = F \qquad \frac{dy}{dt} = -y(1-x) = G$$

a. (3 pts) Find the 3 steady state solutions (i.e. 3 critical points).

b. (2 pts) Find the Jacobian (matrix A) of the system.

a) $x' = F = 0 \Rightarrow y = -2$ OR $y = -x$

$y' = G = 0$

when $y = -2$

$y' = 2(1-x) = 0$

$x = 1$

$\boxed{(1, -2)}$

when $y = -x$

$y' = x(1-x) = 0$

$x = 0, 1$

$x = 0,\ y = 0,\ \boxed{(0,0)}$

$x = 1,\ y = -1,\ \boxed{(1,-1)}$

b) $A = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$

$= \begin{pmatrix} -2-y & -x-2-2y \\ y & -1+x \end{pmatrix}$

$F_x = -(2+y)$

$F_y = -(x+y)-(2+y)$

$G_x = y$

$G_y = -(1-x)$

7. Bonus Question (1 pt) Given a system $\boldsymbol{y}' = \begin{pmatrix} \alpha + 1 & -3 \\ 2 & \alpha + 1 \end{pmatrix} \boldsymbol{y}$. Determine the value(s) of α where the system is ***neutrally stable***.

$$|A - \lambda I| = 0$$

$$(\alpha + 1 - \lambda)^2 + 6 = 0$$

$$\alpha + 1 - \lambda = \pm\sqrt{6}\, i$$

$$\lambda = \alpha + 1 \mp \sqrt{6}\, i$$

neutrally stable

$$\alpha + 1 = 0$$

$$\alpha = -1$$